

CAUCHY'S THEOREM AND CAUCHY'S INTEGRAL FORMULA FOR A-ANALYTIC
FUNCTIONS

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Abstract: This article examines Cauchy's theorem and Cauchy's integral formulas for A-analytic functions. It explores cases where a function is A-analytic and when a function is A-antianalytic.

Keywords: Cauchy's theorem is considered a fundamental theorem in the theory of complex variable functions.

To study the properties of an A-analytic function more deeply, we will examine the following case in this work. The A-analytic function behaves in a specific way within a simply connected domain.

$$\frac{\partial A}{\partial z} = 0$$

1. Theorem (an analogue of Cauchy's theorem). If the function $f(z)$ consists of an A-analytic function in a simply connected domain D , then the value of the integral

$$\int_{\gamma} f(z)(dz + A(z)d\bar{z})$$

along any closed curve γ within the domain D is equal to zero.

This theorem is one of the important theorems in the theory of A-analytic functions.

Proof. We use the well-known Stokes formula for the function $f(z)$ with a continuous derivative in a given domain D

$$\begin{aligned} \int_{\partial D} f(z)(dz + A(z)d\bar{z}) &= \int_D d f(z)(dz + A(z)d\bar{z}) = \\ &= \int_D df(z)(dz + A(z)d\bar{z}) + f(z) d(dz + A(z)d\bar{z}) = \\ &= \int_D \left(\frac{\partial f(z)}{\partial z} dz + \frac{\partial f(z)}{\partial \bar{z}} d\bar{z} \right) (dz + A(z)d\bar{z}) + \int_D f(z) \left(\frac{\partial A(z)}{\partial z} dz + \frac{\partial A(z)}{\partial \bar{z}} d\bar{z} \right) d\bar{z} \end{aligned}$$

Since $f(z)$ is an A-analytic function and $A(z)$ is an anti-analytic function in the domain D ,

$$\frac{\partial f(z)}{\partial \bar{z}} = A(z) \frac{\partial f(z)}{\partial z}$$

$$\begin{aligned} & \int_D \frac{\partial f(z)}{\partial z} (dz + A(z)d\bar{z}) - \int_D f(z) \frac{\partial A(z)}{\partial z} d\bar{z} = \\ & \int_D \frac{\partial f(z)}{\partial z} (dz + A(z)d\bar{z}) \wedge (dz + A(z)d\bar{z}) - \int_D f(z) \frac{\partial A(z)}{\partial z} d\bar{z} \wedge (dz + A(z)d\bar{z}) = \\ & = \int_D \frac{\partial f(z)}{\partial z} (A(z)dz \wedge d\bar{z} - A(z)dz \wedge d\bar{z}) = 0 \end{aligned}$$

the theorem has been proven.

2. Theorem. If the function $f(z) \in C^1(D) \cap C(\bar{D})$ is A-analytic in the domain D , then according to the theorem:

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0$$

Proof. To prove this theorem, let us first consider the case where the curve γ consists of a triangular contour.

Suppose the following:

$$\left| \int_{\partial D} f(z)(dz + A(z)d\bar{z}) \right| = M$$

We will demonstrate that $M = 0$. Let's divide the given triangle by its midpoints and connect these points. As a result of this division, we obtain 4 triangular contours $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.

$$\begin{aligned} \int_{\partial \Delta} f(z)(dz + A(z)d\bar{z}) &= \int_{\partial \Delta_1} f(z)(dz + A(z)d\bar{z}) + \\ &+ \int_{\partial \Delta_2} f(z)(dz + A(z)d\bar{z}) + \int_{\partial \Delta_3} f(z)(dz + A(z)d\bar{z}) + \int_{\partial \Delta_4} f(z)(dz + A(z)d\bar{z}) \end{aligned} \quad (1)$$

Therefore, let the integral of the function $f(z)$ in the triangular region be equal to the number M , which is different from zero.

$$\left| \int_{\partial \Delta} f(z)(dz + A(z)d\bar{z}) \right| = M$$

According to the equation (1)

$$M = \left| \int_{\partial\Delta} f(z)(dz + A(z)d\bar{z}) \right| = \left| \int_{\partial\Delta_1} f(z)(dz + A(z)d\bar{z}) \right| + \left| \int_{\partial\Delta_2} f(z)(dz + A(z)d\bar{z}) \right| + \left| \int_{\partial\Delta_3} f(z)(dz + A(z)d\bar{z}) \right| + \left| \int_{\partial\Delta_4} f(z)(dz + A(z)d\bar{z}) \right|$$

These results indicate that the integral across any Δ_k ($k = 1, 2, 3, 4$) region will not be less than $\frac{M}{4}$ otherwise

$$M = \left| \int_{\partial\Delta} f(z)(dz + A(z)d\bar{z}) \right| < 4 \frac{M}{4} = M$$

i.e. we come up with the meaningless inequality $M < M$. Let this triangular region be Δ_1 , then

$$\left| \int_{\partial\Delta_1} f(z)(dz + A(z)d\bar{z}) \right| \geq \frac{M}{4}$$

Now we divide the region Δ_1 into 4 $\Delta_1^{(1)}, \Delta_1^{(2)}, \Delta_1^{(3)}, \Delta_1^{(4)}$ triangles using the method described above, and among these triangles, we find Δ_2 triangle so that it is

$$\left| \int_{\partial\Delta_2} f(z)(dz + A(z)d\bar{z}) \right| \geq \frac{M}{4^2}$$

We repeat this process indefinitely. As a result, the following sequence of triangles is formed: $\Delta = \Delta_1, \Delta_2, \dots, \Delta_n$

For each Δ_n triangle the following inequality is applicable

$$\left| \int_{\partial\Delta_n} f(z)(dz + A(z)d\bar{z}) \right| \geq \frac{M}{4^n} \quad (n = 0, 1, 2, \dots) \quad (2)$$

The perimeter of the triangle Δ is ℓ , and the perimeters of the contours of the triangles $\Delta_1, \Delta_2, \dots, \Delta_n$ are $\frac{\ell}{2}, \frac{\ell}{2^2}, \frac{\ell}{2^3}, \dots, \frac{\ell}{2^n}$ respectively.

In the sequence of triangles Δ_n , each contour length or perimeter approaches zero at $n \rightarrow \infty$. As a result, there exists a unique point $z_0 \in \Delta_n$, ($n = 0, 1, 2, \dots$) such that, according to the condition, the function $f(z)$ is A-analytic in the domain D at this point. Thus, the function $f(z)$ in the neighborhood of sufficiently small $\{|z - z_0| < \varepsilon\}$ of the point z_0 is expressed as follows

$$f(z) = f(z_0) + \left. \frac{\partial f}{\partial z} \right|_{z=z_0} (z - z_0) + \left. \frac{\partial f}{\partial \bar{z}} \right|_{z=z_0} (\bar{z} - \bar{z}_0) + o(|z - z_0|) \quad (3)$$

The function is not A-analytic in domain D . z also represents the arbitrary neighborhood of a point as follows

$$A(z) = A(z_0) + \left. \frac{\partial A}{\partial \bar{z}} \right|_{z=z_0} (\bar{z} - \bar{z}_0) + o(|z - z_0|) \quad (4)$$

We denote the values of the derivative $\left. \frac{\partial f}{\partial z} \right|_{z=z_0}$ and $\left. \frac{\partial A}{\partial \bar{z}} \right|_{z=z_0}$ as α and β , respectively. Now, using equations (3) and (4), we conclude that the function $f(z)$ is A-analytic.

$$\int_{\partial \Delta_n} f(z) (dz + A(z) d\bar{z}) = \int_{\partial \Delta_n} f(z_0) (dz + A(z_0) d\bar{z}) + o(|z - z_0|)$$

$$dz + \left(A(z_0) + \beta (\bar{z} - \bar{z}_0) + o(|z - z_0|) \right) d\bar{z}.$$

We rearrange the expression under the integral, rewrite the last integral $\int_{\partial \Delta_n} f(z) (dz + A(z) d\bar{z}) = \int_{\partial \Delta_n} f(z_0) dz + \int_{\partial \Delta_n} (A(z_0) + \beta (\bar{z} - \bar{z}_0)) d\bar{z} +$

$$\int_{\partial \Delta_n} [\alpha (z - z_0) + \alpha A(z_0) (\bar{z} - \bar{z}_0)] \beta (\bar{z} - \bar{z}_0) d\bar{z} +$$

$$+ \int_{\partial \Delta_n} [\alpha (z - z_0) + \alpha A(z_0) (\bar{z} - \bar{z}_0)] [dz + A(z_0) d\bar{z}] + \int_{\partial \Delta_n} o(|z - z_0|) dz.$$

The first and third integrals on the right-hand side of the equation are equal to zero. Therefore, in pairs of functions

$$f_1(z) = f(z_0),$$

$$A_1(z) = A(z_0) + \beta (\bar{z} - \bar{z}_0)$$

and

$$f_2(z) = \alpha (z - z_0) + \alpha A(z_0)(\bar{z} - \bar{z}_0)$$

$$A_2(z) = A(z_0)$$

all conditions of Theorem 2 are satisfied. Then, we record the following

$$\int_{\partial\Delta_n} f(z)(dz + A(z)d\bar{z}) = \int_{\partial\Delta_n} [\alpha (z - z_0) + \alpha A(z_0)(\bar{z} - \bar{z}_0)] \beta (\bar{z} - \bar{z}_0)d\bar{z} + o(|z - z_0|)dz.$$

and fix the number $\varepsilon > 0$. Then, for a sufficiently large n we have the following

$$o(|z - z_0|) < \varepsilon d(\Delta_n)$$

Here $d(\Delta_n)$ – diameter of the triangle Δ_n and $d(\Delta_n) = \frac{1}{2^n}$. Hence

$$\left| \int_{\partial\Delta_n} f(z)(dz + A(z)d\bar{z}) \right| \leq |\alpha \beta| \frac{\ell^2}{4^n} (1 + |A(z_0)|) \frac{\ell}{2^n} + \varepsilon \frac{\ell^2}{4^n}$$

$$C \frac{\ell}{2^n} + \varepsilon \frac{\ell^2}{4^n}$$

Here $C = \text{const}$. To consider (2)

$$\frac{M}{4^n} < C \frac{\ell}{2^n} + \varepsilon \frac{\ell^2}{4^n}$$

Or

$$M < \text{Const } \varepsilon$$

Here $M = 0$ as ε is an arbitrary positive number.

We have proven that any triangle Δ in region (Δ, D)

$$f(z)(dz + A(z)d\bar{z}) = 0$$

$\partial\Delta$

Now we show that for any polygon P for the region $(P \quad D)$ the following equation makes sense

$$f(z)(dz + A(z)d\bar{z}) = 0$$

∂P

We will use the general topologic method, i.e.

$$f(z)(dz + A(z)d\bar{z}) = 0$$

γ

The integral of any smooth closed line γ in region D is zero. The theorem has been proven.

We now find an analogue of the Cauchy integral formula for the A-analytic function.

We need the following definition for that.

Definition. A function with the center at the point z_0 is called an A-lemniscate if the polynomial is expressed the following way

$$z : \left| z - z_0 + \int_{z_0}^z \overline{A(\tau)} d\tau \right| < R$$

The parts are determined by $L_R(z_0)$ at the point z_0 .

Note. We will consider the function $K(z) = \frac{1}{z + \int_0^z \overline{A(\tau)} d\tau}$ in the domain D associated with

the unknown polynomial below.

$$z : \left| z - z_0 + \int_{z_0}^z \overline{A(\tau)} d\tau \right| < R$$

Function $A(z)$ in the specified domain D is antianalytic.

Let's assume that, in general, the region D includes the origin. This function has the following conditions

1) $K(z)$ function is an A-analytic function in the domain $D \setminus \{0\}$. In fact the following two make sense

$$\frac{\partial K}{\partial \bar{z}} = - \frac{A(z)}{z + \int_0^z A(\tau) d\tau}$$

and

$$\frac{\partial K}{\partial z} = - \frac{1}{z + \int_0^z A(\tau) d\tau}$$

Here we have the required confirmation from $z = 0$.

2) For the number $\forall \varepsilon > 0$, the integral along the next boundary is related to the part of the A-lemniscate $L_\varepsilon(0) = D$, whose center is at the origin, and is equal to $2\pi i$, i.e.

$$\int_{\partial L_\varepsilon(0)} K(z)(dz + A(z)d\bar{z}) = \frac{d\omega}{\omega} = 2\pi i \quad (5)$$

Proof. We introduce a notation to prove equation (5).

$\omega = z + \int_0^z A(\tau) d\tau$ then $d\omega = dz + A(z)d\bar{z}$ In fact, the following makes sense to be

$$d\omega = \frac{\partial}{\partial z} \left(z + \int_0^z A(\tau) d\tau \right) dz + \frac{\partial}{\partial \bar{z}} \left(z + \int_0^z A(\tau) d\tau \right) d\bar{z} = dz + A(z)d\bar{z}$$

Here, the left side of the integral is equal to (5). Now let's rewrite the expression (5)

$$\int_{\partial L_\varepsilon(0)} K(z)(dz + A(z)d\bar{z}) = \frac{d\omega}{\omega} = 2\pi i$$

Now we express the A-analytic function using the boundary integral in the compact domain.

3. **Theorem.** Let $f(z)$ – be A-analytic function bounded by a smooth line in the domain D and $G = D$. Then, at the point $\forall z \in G$, the function $f(z)$ is expressed as follows

$$f(z) = \frac{1}{2\pi i} \int_{\partial G} f(\zeta) K(\zeta - z)(d\zeta + A(\zeta)d\bar{\zeta}),$$

Here, ∂G – the boundary of the oriented G .

Proof. $\rho > 0$

$$U_\rho = z' : \left| z - z' + \int_{z'}^z \overline{A(\tau)} d\tau \right| < \rho$$

G is a compact region. Let's introduce $G_\rho = G \setminus \overline{U_\rho}$ markup. Then the function is as follows

$$g(\zeta) = \frac{f(\zeta)}{\int_{z'}^{\zeta} \overline{A(\tau)} d\tau}$$

According to Theorem 2, $\overline{G_\rho}$ is an A-analytic function

$$g(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) = 0.$$

∂G_ρ

From here

$$g(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) = \int_{\partial G} g(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) \quad (6)$$

∂G ∂U_ρ

The function $f(z)$ is continuous at the point z . For $\forall \varepsilon > 0$, $\delta > 0$ can be chosen as sufficiently small, for $\rho < \delta$ $|f(z) - f(\zeta)| < \varepsilon$, $\zeta \in \partial U_\rho$ is true. From here

$$\left| f(z) - \frac{1}{2\pi i} \int_{\partial U_\rho} g(\zeta)(d\zeta + A(\zeta)d\bar{\zeta}) \right| =$$

$$= \left| \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{(f(z) - f(\zeta))(d\zeta + A(\zeta)d\bar{\zeta})}{\int_{z'}^{\zeta} \overline{A(\tau)} d\tau} \right| < \frac{1}{2\pi} \varepsilon \cdot 2\pi = \varepsilon.$$

Observations



$f(z) - \frac{1}{2\pi i} \int_{\partial U_\rho} g(\zeta) (d\zeta + A(\zeta)d\bar{\zeta})$ the difference shows that $\rho \rightarrow 0$ approaches zero. From

this we obtain the required equality. Q.E.D.

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