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THE PYTHAGOREAN THEOREM: EXPLORING ITS UNIQUE PROPERTIES.

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Abstract: Over the years scientist, mathematics, and inventors have contributed to the advancement of the Pythagorean theorem. This basic theory lays foundation to everything in our life. This research paper has touched the brief history, tracing its origins from ancient Babylonian and Chinese civilizations, and contextualizing the contributions that led to its formalization by Pythagoras. Beyond historical inquiry, the study outlined three unique properties of the Pythagorean theorem: (1) its membership of every integer in a Pythagorean triple; (2) its nonexistence of a Pythagorean triple consisting entirely of primes; and (3) its beautiful triangle and parity of legs.

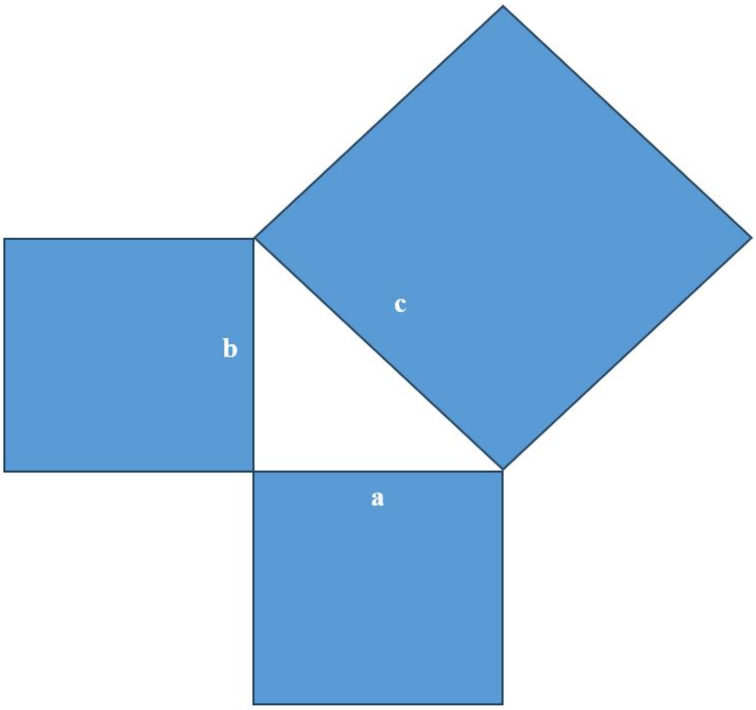
Keywords: Pythagorean theorem, mathematics, scientists, inventors, history, geometry, Pythagoras, Pythagorean Triples.

Introduction

The legendary theory which is familiar to almost all. Pythagorean theory asserts that if the two legs of the right triangle are a and b, and c the length of hypothenuse (the ancient greek name meaning the opposite of the right angle), then the combined areas of the two squares built on the those two legs are equal to the area of the square constructed on the hypotenuse; This fundamental theory was, is, and will be used in various aspects of our life: infrastructure, technology, design, GPS navigation, science, education and research. Bronowski stated that "to this day, the theorem of Pythagoras remains the most important single theorem in the whole of mathematics" (as cited in Agarwal, 2020, p. 3). Similarly, Kaku described it as "the foundation of all architecture: every structure built on this planet is based on it" (as cited in Agarwal, 2020, p. 3).

$$a^2+b^2=c^2$$

(see Figure 1)



For more than 2500 years, the legendary equation a^2+b^2=c^2 has been attributed to Pythagoras. However, the truth goes even beyond the Pythagoras. While the ancient Egyptians applied its practical aspects for construction, historical evidence shows that the core principle of the theorem was already understood by the Babylonians and Chinese centuries before Pythagoras. The Egyptians used this theory practically to build pyramids, even if the theory was not named like that. The construction of pyramids required solid squares. To build this, Egyptians needed to form exact 90-degree angles, which the Pythagorean triplet 3-4-5 helps to create.

This practical Egyptian knowledge finds its theoretical counterpart in Babylonian mathematical texts, which provide the earliest documented evidence of the Pythagorean theorem. In their study of Babylonian mathematics, Smith (2020) describes several tablets related to the Pythagorean theorem, including YBC 7289 and Plimpton 322. Smith provides both historical context and a translated problem from one of the tablets:

We restrict ourselves to the tablets which are related to the Pythagoras' theorem, in particular the tablets YBC 7289 (from the Yale University collection), Plimpton 322 (from the Columbia University collection), the more recently excavated Susa tablet (discovered at Shoosh, Iran) and Tell Dhibayi tablet (discovered near Baghdad, Iraq). All these tablets are dated between 1900 BCE and 1600 BCE. However, before we proceed further, here is a translation of a problem and its solution inscribed on a Babylonian tablet kept at the British Museum, London.

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4 is the length and 5 is the diagonal. What is the breadth? Its size is not known. 4 times 4 is 16, 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9, 3 is the breadth. (Smith, 2020, p. 4)

To confirm the fact that Babylonians were quite familiar with geometry except Pythagorean triples, we look at the tablet YBC 7289. The diagram on this tablet is shown in Figure 1.

As Rahul Roy (2003) explained:

On one side of the square is inscribed 30, while on the horizontal diagonal are inscribed two numbers 1 24 51 10 and 42 25 35. Recalling that the Babylonians had a sexagesimal system3 and assuming that the space between the digits represent position values of the (sexagesimal) digits and also assuming that the first number is 1.245110(mod 60) and the second number is 42.2535422535(mod60), then translating these numbers gives 1.245110(mod10) = 1.41421296(mod10) and 42.253542 2535(mod60) = 42.4263888(mod 10).
Although this confirms that the Babylonians were familiar with the geometry behind Pythagorean triplets, it raises the question as to how did they compute √2 so accurately. (pp. 33–34)

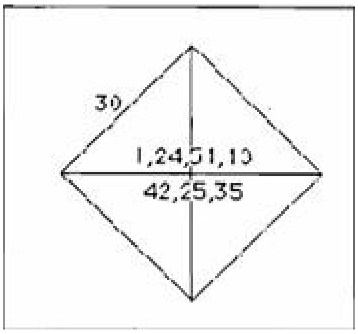

Figure 1. Diagram on the tablet YBC 7289

Chinese culture has long been associated with mathematical developments. The *Zhoubi suanjing*, the oldest known Chinese mathematical work, exemplifies this rich heritage (Gustafson, 2012, p. 207). Though the author of this work is unknown, it is believed to be sometime between 100 BC and 100 AD. Scholars note that the contents were likely from a much earlier period (Li, 1987, as cited in Gustafson, 2012, p. 207). This ancient text consists of two sections: mathematics and astronomy. To understand the second section (astronomy) explanations, mathematics in the first section is important, particularly the discussion of right triangle theory - known is the West as the Pythagorean theorem but referred to as the *gou-gu* theorem in ancient Chinese literature (Gustafson, 2012, pp. 207-208).

Pythagorean theorem is one of the most popular theorem in geometry, containing more proofs than any other theorem in mathematics. In 1927, mathematics teacher from Ohio Elisha S.Loomis (1852-1940) wrote 370 different proofs in *The Pythagorean Preposition*: Algebraic (109); Geometric (255); Quaternionic (4); and those based on mass and velocity, Dynamic (2). This simple theorem got high praise in 2004 from the readers of *Physics World* magazine, holding the forth place out of 20. (Maor, 2007)

Unique Properties of the Pythagorean theorem

1. Membership of every integer in a Pythagorean triple

Statement.
For every integer $n > 2$, there exist integers x and y such that (n, x, y), in some order, forms a Pythagorean triple; that is, $n^2 + x^2 = y^2$ or $x^2 + y^2 = n^2$.

Proof (Constructive).
Case A: n odd.
Let n be odd. Define:
$x = (n^2 - 1)/2, \quad y = (n^2 + 1)/2.$
Since $n^2 \pm 1 \equiv 0 \pmod 2$). Both x and y are integers. A direct computation shows that:
$n^2 + x^2 = y^2$.
Thus (n, x, y) forms a Pythagorean triple.

Case B: n even.
Let $n = 2m$ with $m \geq 2$. Define:
$x = m^2 - 1, \quad y = m^2 + 1.$
Then:
$x^2 + n^2 = y^2$.
Thus (x, n, y) forms a Pythagorean triple.

Therefore, every integer $n > 2$ appears in some Pythagorean triple.

Examples:
$n = 3 \rightarrow (3, 4, 5)$
$n = 4 \rightarrow (3, 4, 5)$
$n = 5 \rightarrow (5, 12, 13)$
$n = 10 \rightarrow (10, 24, 26)$.

2. Nonexistence of a Pythagorean triple consisting entirely of primes

Statement.
There are no three prime numbers p, q, r such that $p^2 + q^2 = r^2$.

Proof.
Since 2 cannot be in some Pythagorean triplet, p and q are odd primes, then $p^2 \equiv q^2 \equiv 1 \pmod 4$, hence:
$p^2 + q^2 \equiv 2 \pmod 4$.

Now, observe that a perfect square modulo 4 can only be 0 or 1**.** Any integer number can be represented as 2*k and 2*k+1, where k is also integer.

$(2k)^2 = 4k^2 \Rightarrow 4k^2 \equiv 0 \pmod 4$.

$(2k+1)^2 = 4k^2 + 4k + 1 \Rightarrow (2k+1)^2 \equiv 1 \pmod 4$.

Therefore $p^2 + q^2$ cannot be a perfect square.

3. The beautiful triangle and parity of legs.

Statement:

The radius of the inscribed circle of any right triangle with integer sides is always an integer.

Proof.
Let the sides of a right triangle with integer lengths be a, b, and c, where c is the hypotenuse. The radius r of the inscribed circle in a right triangle is given by

$r = (a + b - c) / 2$.

Hence, it is sufficient to prove that $a + b - c$ is even for all integer-sided right triangles.

Assume that (a, b, c) is a primitive Pythagorean triple, that is, $\gcd(a, b, c) = 1$ and $a^2 + b^2 = c^2$.

If both legs a and b were odd, then $a^2 \equiv b^2 \equiv 1 \pmod 4$, so $a^2 + b^2 \equiv 2 \pmod 4$, which is impossible since no perfect square is congruent to 2 modulo 4. Thus, both legs cannot be odd.

In a primitive triple, the two legs cannot both be even either, since that would imply a common factor of 2.
Therefore, exactly one leg is even and the other is odd.

Consequently, $a^2 + b^2$ (even + odd) is odd, which means c^2 is odd and hence c is odd. Thus, in every primitive Pythagorean triple, one leg is even, the other is odd, and the hypotenuse is odd. Therefore $a + b - c$ is even.

Examples:

1. Triangle: (3, 4, 5)

   - Legs: 3 (odd), 4 (even)

   - Hypotenuse: 5 (odd)

   - Inscribed circle radius: $r = (3 + 4 - 5)/2 = 1$

2. Triangle: (5, 12, 13)

   - Legs: 5 (odd), 12 (even)

   - Hypotenuse: 13 (odd)

   - Inscribed circle radius: $r = (5 + 12 - 13)/2 = 2$

3. Triangle: (7, 24, 25)

   - Legs: 7 (odd), 24 (even)

- Hypotenuse: 25 (odd)
- Inscribed circle radius: r = (7 + 24 - 25)/2 = 3

Conclusion

1. The history of the Pythagorean theorem dates back to the times of ancient Egypt, Babylon, China, and obviously Greece, each of these contributing to the development and discovery of it. Ever since those times the world-renowned theorem has long been useful to humans, assisting in various aspects of their lives such as architecture, science, navigation, and etc. Humans have even dedicated their lives to counting the numerous ways of proving the accuracy of the Pythagorean theorem. While everyone is familiar with this theorem, not all, however, realize the three unique properties it holds. This research listed these properties such as membership of every integer in a Pythagorean triple, nonexistence of a Pythagorean triple consisting entirely of primes, and the beautiful triangle and parity of legs. As the research went over these properties in great detail, people – both professionals and simple folk – can utilize the research for their personal needs.

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